

New Interactive Animations for Analysis of Stress, Strain, and Flow

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Abstract: In geological investigations, stress, strain, and flow in structures are commonly analyzed with the aid of Mohr Diagrams. This paper presents new alternative constructions that build on the techniques familiar to structural geologists from the use of stereonet. The constructions avoid many of the complexities of previous methods, such as off-axis Mohr circles and poles, and present vector data (such as the stress vector acting on an inclined plane) in the correct spatial orientation. The relationship between new and existing constructions is easily demonstrated. The constructions are animated in the electronic version of this paper and may be viewed with a web browser using Macromedia Flash 5.

Introduction

For more than a century, geologists have used theoretical and practical analysis of stress, strain, and flow to help in the understanding of tectonic structures (e.g., Means 1976, 1990). Among the graphical tools currently available are the Mohr constructions for stress, infinitesimal strain and reciprocal quadratic elongation. Modifications to these constructions for two-dimensional stretch and flow have made them more broadly applicable to general deformations and more powerful (Passchier and Urai 1988, Bons and Urai 1992) but this also has added to their complexity (De Paor and Means 1984). Whilst Mohr constructions greatly aid in the visualization of deformation and the discovery of numerical solutions to structural problems, the burden of learning the mathematical derivations, even of the simpler classical constructions may turn students off structural studies at an early stage. Many geochemists and paleontologists admit that they chose their career paths to avoid things like Mohr Diagrams!

New Construction

This paper presents a new approach which builds on the geologist's existing familiarity with stereographic projection (e.g., Davis and Reynolds 1996, p. 691) and avoids some of the complexity associated with previous methods. The construction, which the author's students have dubbed "Declan's Doughnut," for reasons that will become apparent, improves on both classical Mohr constructions and modern variants in that a vector is presented in its correct spatial orientation; in contrast, other plots relate stress or strain magnitudes to spatial orientations via complex constructions.

System Requirements

To view the animations in the electronic version of this paper, you will need a Java-enabled, version 4 or higher web browser. You also must have version 5, or higher, of Macromedia's Shockwave Flash™ plugin which is available for Windows, Mac, Linux and Solaris. The vast majority of web browsers have Flash installed (Macromedia claim a market penetration in excess of 90%) but most utilize version 4 which permits viewing but not full interactivity.

To download and install Flash version 5, go to <http://www.macromedia.com/shockwave/download>. If the animations are still not working after installation, check your browser's settings to ensure that the plugin is indeed in the current browser's plugin folder, and that it is designated as the "helper application" for files with the suffix ".swf" and the MIME type "application/x-shockwave-flash."

The Underlying Grid

The new constructions employ a polar grid of the type used for the plotting of linear rose diagrams or scatter graphs of vector data (Fig. 1). The difference between this grid and a polar stereonet such as the Lambert Projection lies in the constant spacing of the concentric circles and the arbitrary outer limit here chosen as 10 bold units.

To facilitate scaling operations, a net of 10 cm radius with concentric circles drawn bold every 1 cm and light every 2 mm is used here. (The actual units are not specified - they could be Megapascals, or pounds per square inch for stress analysis, pure numbers in the case of strain, or rates in the case of flow). Angles are marked lightly at 2-degree intervals and heavily every 10-degrees, over all but the

innermost part of the grid. The source is a standard .gif file call DonutPaper.gif. You may download a zipped (PC) or stuffed (Mac) version for printing and reproduction. Alternatively, just drag Fig. 1 from your browser window to the computer desktop.

The Simplest Stress Construction: Uniaxial Stress: Animated Version

To begin, let us consider the simple problem of determining the total stress σ_t on an arbitrary inclined plane under a uniaxial principal stress, say $\sigma_I = 200$ MPa (the subscript t is used because σ_t refers to the total load on a plane per unit area and generally has both normal and shear stress components, σ_n and σ_s). The problem is solved by an interactive Flash animation, a snapshot of which is illustrated in Fig. 2. A circle is drawn with a diameter representing 200 MPa. A bold red line represents a vertical stress vector. Solid and hashed green lines of arbitrary length represent, respectively, the normal and trace of the plane on which that stress acts. If you drag the computer mouse around the circle, the inclined plane tracks the mouse movement and the red line represents the total stress, σ_t , as a function of pole orientation, θ_n . Toggling the show/hide button reveals the normal and shear stress components which are drawn in magenta. Note that the orientation of the stress vector in this case remains constant; only the stress magnitude changes with orientation of the plane of application. For simplicity, the maximum stress is chosen vertical in the case illustrated; other orientations are simply entered into the θ_0 text field.

The Hand-drawn Construction for Uniaxial Stress

To create an equivalent construction using pencil and tracing paper, print Fig. 1 onto plain paper or acquire a polar grid and ruler, and then follow these steps:

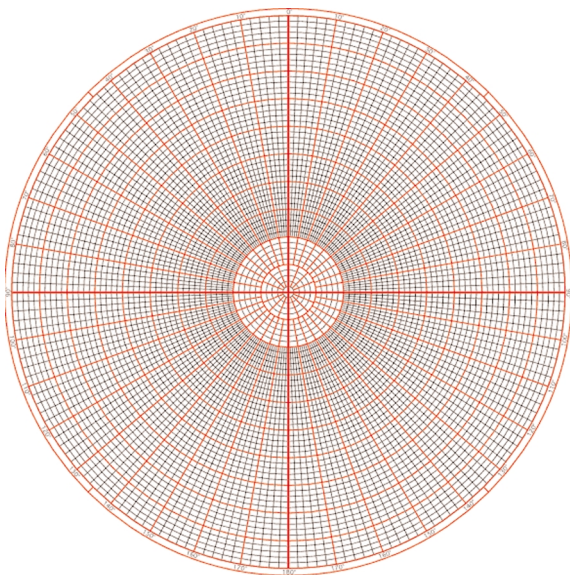


Figure 1: The Polar Grid.

- (i) Mark a point O_p on the perimeter of the grid at zero degrees which we will call the "Origin of Planes."
- (ii) Mark a second point O_s , also on the grid, at 180 degrees, which we will call the "Origin of Stresses."
- (iii) Place a tracing overlay on the grid and draw an arbitrary radius to represent the normal to a plane of interest and a hashed tangent to represent the trace of the plane. The lengths of these radial and tangential lines have no numerical significance; they just mark directions. At the outset, the tracing overlay should be oriented so that these two lines intersect at the origin of planes, O_p .
- (iv) Let the *diameter* of the net represent the principal stress $\sigma_I = 200$ MPa. So, for example, if the net's diameter is 20 cm as in Fig. 1, then every bold (centimeter) division corresponds to 10 MPa and every smaller (2mm) division represents 2 MPa.
- (v) To determine the stress σ_t on a plane whose normal subtends an angle θ_n clockwise from vertical, rotate the tracing overlay clockwise through the angle θ_n from the origin of planes, O_p , then count an angle $-\theta_n$ counterclockwise from the origin of stresses, O_s (this mark is on the grid, not the overlay, so it does not rotate), and join the resultant points.

Step (v) is a rather roundabout way of drawing a vertical line! However, it introduces the general procedure that must be followed in the more complicated cases that follow.

The line thus drawn represents the stress vector σ_t in magnitude *and* orientation. Its shear and normal stress components, σ_s and σ_n , may be found by casting perpendiculars onto the trace of the plane and its normal, and measuring the components with a ruler scaled to 1 MPa per millimeter in this case. Obviously, these stress components are also shown in their correct spatial orientation.

Generalization to cases of non-vertical principal stress is trivial; you simply rotate the two origins, O_p and O_s , and the tracing overlay, to the desired inclination. Using the

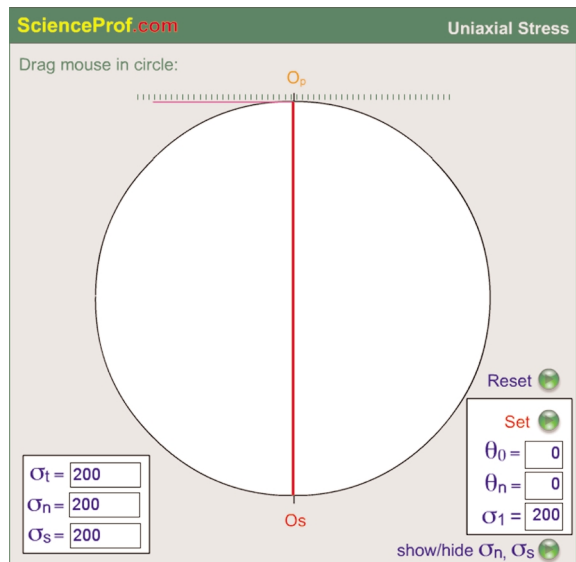


Figure 2. Animation for solving uniaxial stress problems.

Flash animation, you enter a value for θ_0 and click the *set* button. Unlike the classical Mohr construction, there is no complicated procedure for relating data in geographic space to data in stress space.

Slip on Preexisting Failure Plane

The uniaxial stress construction is immediately ready for practical application. Let the angle of sliding friction for slip on a preexisting fracture or fault plane be $\psi = 20$ degrees (correspondingly, the *coefficient of sliding friction* is $\mu = \tan\psi$). We can shade an area analogous to a pre-fractured Mohr-Coulomb envelope, shown in green in Fig. 3. In the hand-construction, this shading is applied on the overlay so that it rotates with the plane's trace and normal. Using the Flash animation linked to Fig. 3, enter the desired angle of friction in the ψ text field and press the set button. Then either drag the mouse to change the plane's orientation or enter its dip in the θ_n text field and click the set button. As long as the red stress vector lies in the green stable field, a block resting on the plane is stable. As soon as the dip angle exceeds the angle of friction, the block slips. [Astute students may question why the block does not accelerate. This is mainly due to author's laziness but also reflects the block's terminal velocity under frictional sliding. The block does slide faster at steeper dips but does not fall off after 90 degrees; no useful purpose, beyond entertainment, would be served by such an animation enhancement.]

By reducing the value of the principal stress in the σ_1 text field, you can powerfully convey the fact that, in this cohesionless scenario, sliding can occur no matter how small the shear stress magnitude, provided the shear to normal stress ratio exceed the coefficient of friction, μ .

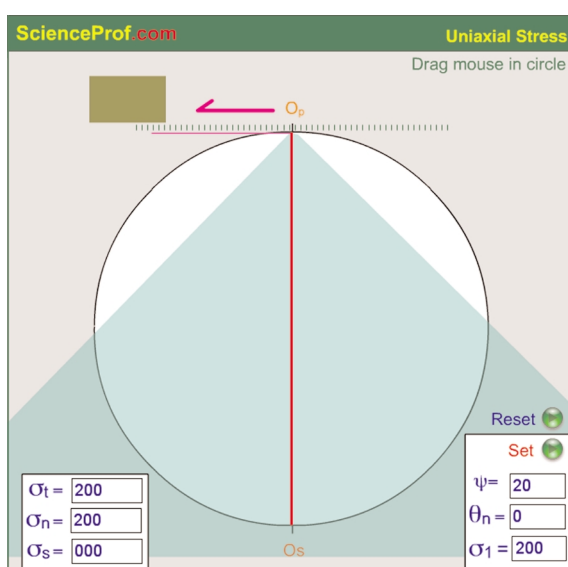


Figure 3. Flash animation of the failure criterion for slip on a preexisting plane.

The Construction for Two Stresses of Like Sign

A more general case is represented in Fig. 4, where two principal stresses, σ_1 (vertical) and σ_2 (horizontal), are either both compressive or both tensile. In this case, two circles are drawn, one with diameter $\sigma_1 + \sigma_2$ and the other with diameter $\sigma_1 - \sigma_2$. The area between these circles is called the "doughnut" (a.k.a. "donut!")

[In the hand-drawn implementation, any suitable scale factor may be used to convert radii from centimeters to Megapascals and *vice versa*. It is not necessary to use the full diameter of the grid if that results in tedious scale conversions. For example, if $\sigma_1 = 120$ MPa and $\sigma_2 = 60$ MPa then $(\sigma_1 + \sigma_2) = 180$ MPa, so a scale of 10 MPa = 1 cm is appropriate, giving an outer circle of 18 cm diameter that easily fits within the 20 cm diameter grid of Fig. 1. To this scale, the inner circle is $(\sigma_1 - \sigma_2) = 60$ MPa = 6 cm.]

The origin of planes, O_p , is marked on the underlying grid at an arbitrary point on the outer edge of the doughnut and the origin of stresses, O_s , is marked also on the underlying grid at the diametrically opposite point on the inner edge of the doughnut (Fig. 4). The distance $O_p - O_s$ is thus equal to σ_1 and the radial width of the doughnut from inner to outer edge is σ_2 .

To determine the stress acting on an arbitrary plane, you follow a similar procedure to the uniaxial case:

- Rotate the tracing overlay with its plane trace and normal axes through the arbitrary angle, say $\theta_n = 26^\circ$, from the origin of planes, O_p .
- Mark off $-\theta_n = -26^\circ$ from the origin of stresses O_s around the inner circle, and connect these points to obtain the total stress, $O\tau$.

This time, the construction is not an unnecessarily obscure way of drawing a vertical line; rather, the stress vector is oblique to the vertical and horizontal for all but the principal values. As before, all stress vectors and planes on which they act are shown in their correct spatial orientations.

There is some danger of accidentally counting off a 26° angle rather than -26° on the inside of the doughnut. However, this error is easily spotted; the result is always a vector with the magnitude of the principal stress, $|\sigma_1|$. Most students will quickly realize that they are doing something wrong! In contrast, confusion of sign conventions in Mohr constructions are rampant and difficult to spot, especially since angles measured on Mohr circles are also doubled. The animation in Fig. 4 also includes an option to show the stress ellipse centered on the outer margin of the doughnut. By dragging the mouse around, you quickly get to appreciate the fact that the ellipse represents the locus of all total stress vectors.

General Failure Envelope

To illustrate the general Mohr-Coulomb type failure criteria on the new constructions, an area is shaded green as in Fig. 5; however, the shaded outline is now parabolic,

representing the transition from brittle to ductile deformation mechanisms with depth in the earth's crust, and it extends past the trace of the plane, representing material with a certain tensile strength due to its cohesiveness (Fig. 5a).

Whereas the influence of pore fluid pressure is usually represented by drawing different Mohr circles for stress and effective stress, it makes more sense, following Bayly (1991) to shift the Mohr envelope instead. This is implemented in the animation linked to Fig. 5. When pore pressure is applied, the shaded area shifts away from the plane's trace, reducing the extend of the stable region. As the mouse is dragged around the doughnut, there are some directions in which the stress vector falls outside the shaded stable envelope. Pore fluid pressure is applied in the animation linked to Fig. 5 by clicking the lower right button.

The Construction for Two Stresses of Opposite Sign

When the principal stresses differ in sign (one being compressive, the other tensile), their sum is numerically less than their difference. In this case, therefore, the origin of planes, O_p , lies on the inner boundary of the doughnut and the origin of stresses, O_s , lies on the outside (Fig. 6). If you apply the same rules of construction as before, counting degrees clockwise from O_p and counterclockwise from O_s , or *vice versa*, the construction yields valid results. Now there are two special conjugate orientations in which the normal component of stress drops to zero; in these orientations, the total stress vector is parallel to the trace of the plane and is equal to the shear stress component. These cases correspond to the points where the Mohr circle intersects the shear stress axis of a conventional Mohr

Diagram and they divide the stress ellipse into fields of compression and tension, as color-coded in the animation.

Limitations

The constructions presented thus far are ideal for determining the stress state given fixed values of the principal stresses. However, there are many situations in which it is desirable to examine variations in the stress tensor with time and for these, the classical Mohr construction is still superior; therefore, it is important to be able to transfer from one construction to the other.

The relationship between the doughnut construction and the Mohr Diagram could not be simpler to illustrate, since the inner circle of the doughnut is a rotated version of the Mohr circle; all you need do is lift the tracing overlay off the grid and reorient it with the plane's normal and trace oriented as abscissa and ordinate (Refer back to Fig. 4). Note the relationship between the Mohr circle's poles and the two origins of the doughnut diagram, O_p and O_s .

The role of the traditional Mohr Diagram can be enhanced with the aid of animation, as illustrated in the Flash movies linked to the caption of Fig. 7. This is particularly true at sub-critical stress states early in a deformation history, which are difficult to illustrate with overlapping series of circles, and at elevated confining pressures, where rapid ductile strain rates lead to a dissipation of the applied stress difference. In Fig. 7a, a specimen (blue) is subjected to increased confining pressure in stage (1), represented on the Mohr diagram by a point (circle of zero radius) that moves across the normal stress axis. In stage (2), an axial load is applied via the red and black pistons, creating a differential stress represented by the radius of the growing Mohr circle. In stage (3), the specimen fails along a fracture plane (dash yellow line)

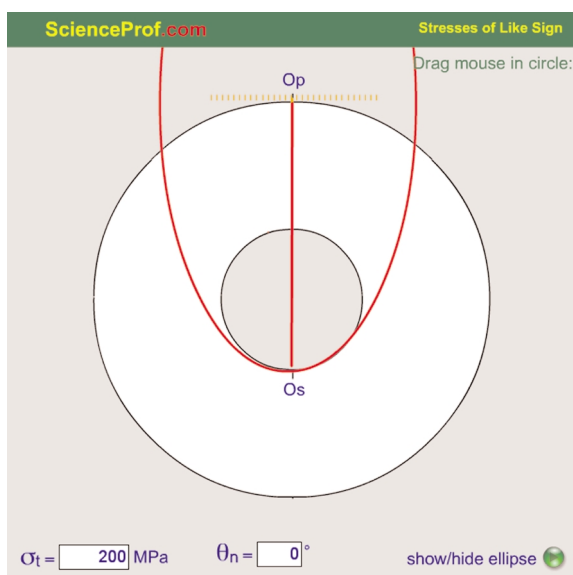


Figure 4. Construction for stresses of like sign. See text for explanation.

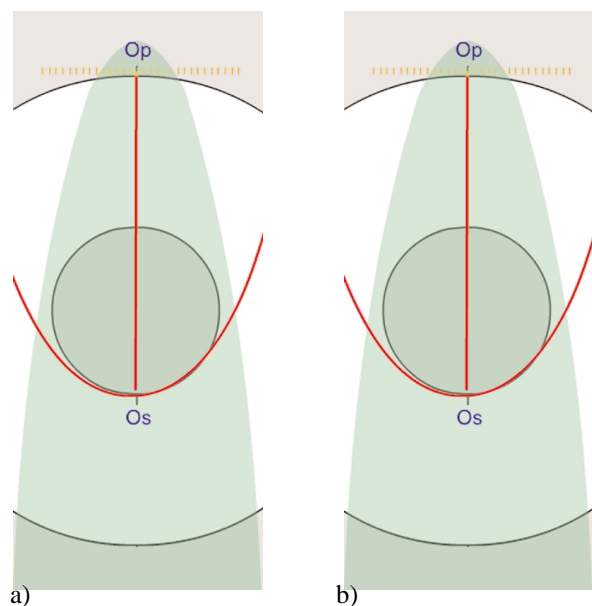


Figure 5. Parabolic failure envelopes for a) total and b) effective stress.

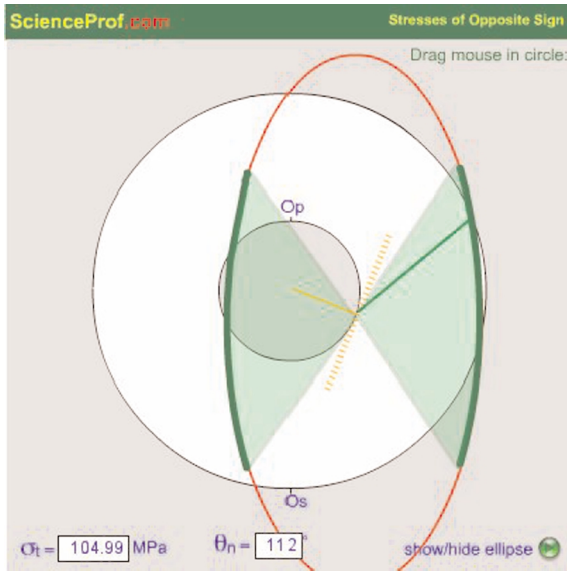


Figure 6. Construction for stresses of unlike sign. The green shaded sectors of the stress ellipse represent the field of tension in this case.

determined by the point of tangency of the Mohr circle and envelope. The magnitudes of the normal and shear stresses at the moment of failure are indicated by the red dot in stress space. Whether the plane shown here or its conjugate (reflected through the vertical) becomes the site of specimen failure is a matter of chance.

Figure 7b shows a similar experiment in which the specimen yields just before failing, whereas in Fig. 7c, the period of ductile flow after yielding is protracted. The differential stress that the specimen can maintain (which equals the diameter of the Mohr circle) is a function of strain rate, since strain dissipates the applied load by continuous flow on ductile shear zones (dashed yellow lines). The orientations of the ductile shear zones on which the specimen deforms and the stresses on those shear planes (yellow spots) are determined by "Bayly Curves" (1991 - dashed) that contour strain rate, as indicated by the "strain rate -->" label. These curves are a function of the material and may be straight, or even concave.

The Strain Construction

The construction for strain is similar to that for stress except that the origins are renamed "origin of lines", O_L , and "origin of strains", O_S , respectively, and the deformation generally includes a component of rigid rotation. Figure 8 shows a screen shot from an animation of the case of simple shear.

To construct a diagram for simple shear by hand:

- (i) draw an inner circle with diameter γ equal to the shear strain,
- (ii) draw a tangent of unit length in the direction of slip and label its ends O_L and O_S , as in Fig. 8,
- (iii) draw an outer circle passing through the tangent, O_L .

Now you are ready to determine the stretch and rotation of an arbitrary line, say one at -32° (counterclockwise) from

the origin of lines, O_L . Analogous to the case of stress, rotate the tracing overlay through an angle of -32° and set off an angle $+32^\circ$ clockwise from the origin of strains, O_S , yielding the red line of Fig. 8. This has the deformed length and orientation of the initial orange line. For reference, equivalent off-axis Mohr circle constructions are inset in the animation.

The Flow Construction

Whilst the final state of deformation of a rock is of potentially great interest, kinematic analysis is essential for a full understanding of structural evolution. Fabrics in mylonites are particularly strongly controlled by the kinematic character of the flow regime. An animated flow construction for progressive pure shearing is linked to Fig. 9. This is equivalent to an extreme case of the construction for stresses of unlike sign (Fig. 6) in which the circle representing the sum of the principal vectors shrinks to zero, because they are equal but opposite, and the flow is represented by a circular ellipse with axes of opposite sign (Fig. 10). Nevertheless, the construction rules are unchanged. To determine the flow vector for a particular direction, the tracing overlay is rotated through the desired angle about the circle on which the origin of directions, O_d , resides (i.e. the one that has shrunk to a point) and an equal but opposite angle is set off from the origin of flow, O_f . The origins are renamed so that it is immediately clear from the labels which is a construction for stress, strain, and flow.

Conclusion

All tensor phenomena in nature can be represented in a variety of ways. A purely algebraic representation using matrices suffices in theory but, for most people, visual queues aid in the discovery of solutions to problems. The stress, strain, and flow ellipses (/ellipsoids) are useful for illustration and visualization purposes but they are difficult to plot and measure in quantitative studies. Graphs combine the qualities of visualization and measurement but their effectiveness is reduced when the relationship between the space of plotted values and geographic space becomes obscure, as is certainly the case for the classical Mohr

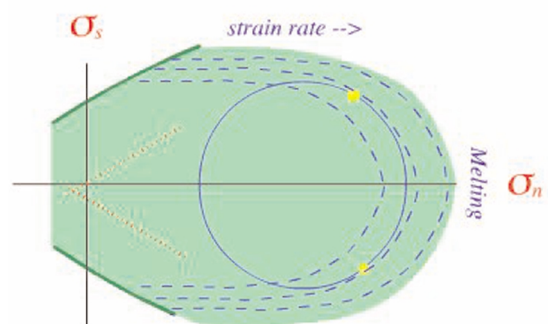


Figure 7. Animation of Mohr Envelope Constructions for (a) brittle deformation, (b) yielding failure, and (c) ductile shear.

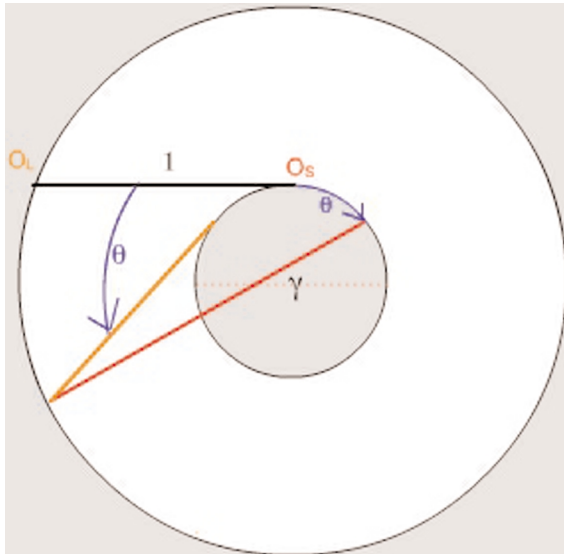


Figure 8. Construction for simple shear strain.

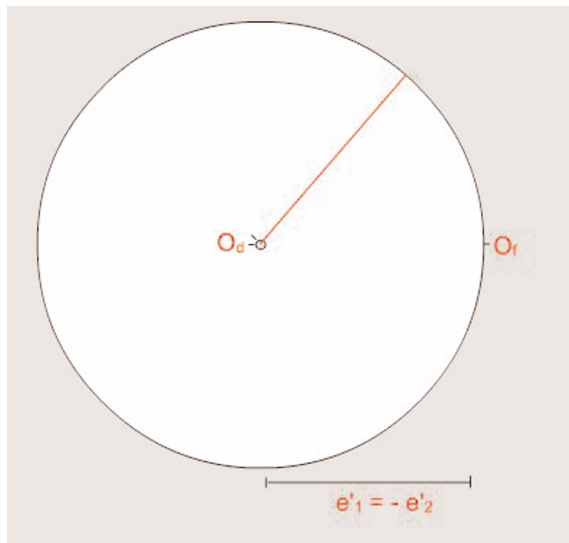


Figure 9. The construction for flow (a.k.a. velocity gradients).

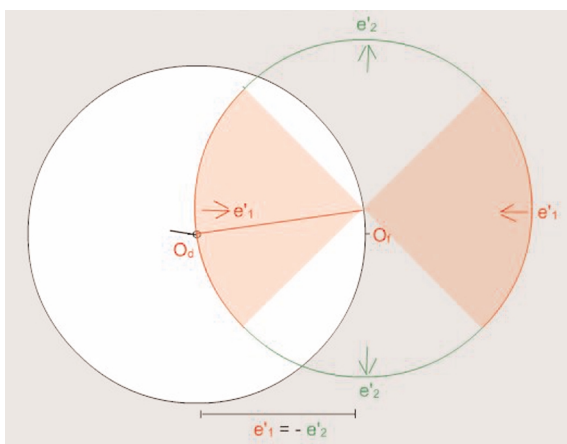


Figure 10. Flow ellipse for progressive pure shearing. The red shaded area represents flow vectors with a component of shortening. The black tick marks the direction to which the red flow vector applies.

construction. The new doughnut construction presents the same data in a more easily visualized format. The plot has the major advantage that it displays vector data in their correct spatial orientations. And there is an added advantage that it builds on the orientation net skills that all students of structural geology acquire at an early stage. It obviates the need for complex constructions involving Mohr circle poles and off-axis circles.

The method is implemented in the on-line version of this paper using computer animation but the construction is also suitable for traditional pencil-and-tracing-paper applications in the teaching laboratory. Indeed, it is important for students to practice all such methods by hand; the role of animation is to aid visualization and rapid acquisition of a basic understand, and also to provide convenience and accuracy after the hand method has been mastered. The author's recent teaching experience (ES71 - Tectonics and Structural Geology - at Harvard University, and ES301 - Structural Analysis - at Boston University) suggests that the new approach saves at least one hour of lecture time and four hours of lab. Seemingly trivial details, such as the fact that, in the stress construction, compression is plotted downwards, dextral shear to the right, and sinistral shear to the left, can add up to a significant saving in comprehension time and effort.

Of course, there are cases where both the classical and modern Mohr constructions are desirable, in which cases the polar plots are converted to the Mohr reference frame simply by lifting the tracing overlay off the polar net and orienting it horizontally.

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